

Non-vanishing of Artin-twisted L -functions of Elliptic Curves

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Abstract: *Let E be an elliptic curve and ρ an Artin representation, both defined over \mathbb{Q} . Let S be a finite set of primes, including a prime p at which E has good reduction. We prove that there exists an infinite set of Dirichlet characters χ , unramified outside S , such that the Artin-twisted L -values $L(E, \rho \otimes \chi, \beta)$ are non-zero when β lies in a specified region in the critical strip (assuming the conjectural continuations and functional equations for these L -functions).*

1 Introduction

An Artin representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is defined to be a finite-dimensional complex representation which factors through a finite extension K/\mathbb{Q} . If E is an elliptic curve over \mathbb{Q} , there exists an L -series $L(E, \rho, s)$ associated to E and ρ , which appears naturally as a factor of the L -function of E over K .

The function $L(E, \rho, s)$ is defined by an Euler product which converges only on the region $\text{Re}(s) > 3/2$, but conjecturally it may be analytically continued to the entire complex plane and will satisfy a functional equation (we give further details in §3). The purpose of this paper is to show that, once we assume the continuation and functional equation for these L -series, we can prove the non-vanishing of infinitely many Dirichlet twists of $L(E, \rho, s)$ in a certain region in the critical strip. The precise statement of our main theorem is as follows.

Theorem 1.1. *Let S be a finite set of primes, containing a prime p such that E has good reduction at p . Suppose further that the L -function $L(E, \rho \otimes \chi, s)$ satisfies the conjectural analytic continuation and functional equation for all Dirichlet twists χ . Then, for any β in the critical strip $\{s \in \mathbb{C} : 1/2 \leq \text{Re}(s) \leq 3/2\}$ satisfying*

$$\text{Re}(\beta) \notin \left[\frac{1}{2} + \frac{2}{2 \dim \rho + 1}, \frac{3}{2} - \frac{2}{2 \dim \rho + 1} \right]$$

we have $L(E, \rho \otimes \chi, \beta) \neq 0$ for infinitely many Dirichlet characters χ which are unramified outside S .

This theorem is motivated by the many results in the literature on the non-vanishing of twists of L -functions; for example those of Rohrlich [10, 11], Barthel and Ramakrishnan [1], Friedberg and Hoffstein [5], and Luo [8]. To prove Theorem 1.1 we follow a standard technique of using the approximate functional

equation and averaging over twists. In particular we follow the method of Luo from [8] (previously used by Iwaniec in [6]).

We conclude the introduction with the following remark: suppose ρ is 2-dimensional, irreducible and odd (that is, the determinant of complex conjugation is -1). Under these assumptions, except for certain icosahedral examples of ρ , it is known that ρ is equivalent to the representation given by a weight 1 newform (this is proved by Buzzard, Dickinson, Shepherd-Barron and Taylor in [3]). As we also know that E is modular by the work of Wiles et al in [14] and [2] we may write $L(E, \rho, s)$ as a Rankin convolution of modular forms. The functional equation and continuation are proven for any Dirichlet twist of such an L -function, so Theorem 1.1 holds unconditionally in this case.

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2 Conductors of twisted Artin representations

In our main theorem, we impose no restrictions on the ramification of ρ at the primes in S . For this reason we first prove a result on the Artin conductors of the twists $\rho \otimes \chi$; we begin with a preparatory lemma on ramification groups.

Lemma 2.1. *Let p be a prime. Let L/\mathbb{Q}_p be a finite Galois extension of local fields, and suppose*

$$\phi : \text{Gal}(L/\mathbb{Q}_p) \rightarrow A$$

is a homomorphism of groups. Let $K_n = \mathbb{Q}_p(\mu_{p^n})$ and regard ρ as a homomorphism $\text{Gal}(LK_n/\mathbb{Q}_p) \rightarrow A$ by extension through the quotient map. Then there exists a fixed integer i_ϕ such that

$$G_i(LK_n/\mathbb{Q}_p) \subseteq \ker \phi$$

for all $i \geq i_\phi$, for all sufficiently large n . Here $G_i(LK_n/\mathbb{Q}_p)$ denotes the i -th ramification group of the extension LK_n/\mathbb{Q}_p .

Proof. Let $H = \text{Gal}(LK_n/L)$, so that $H \subseteq \ker \phi$. Quoting [9, Chapter II, §10] we have the formula

$$\frac{G_i(LK_n/\mathbb{Q}_p) H}{H} = G_t(L/\mathbb{Q}_p)$$

where $t = \eta_{LK_n/L}(i)$. Here, $\eta_{LK_n/L}$ is the function which defines the upper numbering for the ramification groups of LK_n/L (see [9]). Let t' be the some integer such that $G_{t'}(L/\mathbb{Q}_p) = \{1\}$. We will show that there exists a fixed i_ϕ such that $\eta_{LK_n/L}(i) \geq t'$ when $i \geq i_\phi$, for all sufficiently large n . This will establish the claim, as the formula above implies $G_i(LK_n/\mathbb{Q}_p) \subseteq H \subseteq \ker \phi$ for such i .

By definition, we have

$$\eta_{LK_n/L}(i) = \frac{1}{g_0}(g_1 + g_2 + \cdots + g_i)$$

where we write $g_i = |G_i(LK_n/L)|$. Enlarging L if necessary, assume that $\mathbb{Q}_p(\mu_{p^\infty}) \cap L = K_m$. Suppose we have $p^s - 1 \geq i \geq p^{s-1}$, where $n \geq s \geq m$. One checks that

$$G_i(LK_n/L) \supseteq \text{Gal}(LK_n/LK_s)$$

so for such i we have $g_i \geq p^{n-s}$. We also observe that $g_i = p^{n-m}$ for $p^{m-1} \geq i \geq 0$. Therefore we have

$$\begin{aligned} \eta_{LK_n/L}(p^s - 1) &\geq \frac{1}{g_0} \sum_{i=1}^{p^{m-1}-1} g_0 + \frac{1}{g_0} \sum_{t=m}^s p^{n-s}(p^s - p^{s-1}) \\ &= p^{m-1}(1 + (s-m)(p-1)) - 1. \end{aligned}$$

We may choose s to make this quantity as large as we like, independently of n (provided n is sufficiently large), which completes the proof. \square

Now let ρ be an Artin representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The global Artin conductor associated to ρ may be given as a product

$$N_\rho = \prod_{p \text{ prime}} p^{n_p(\rho)}$$

where the local conductors $n_p(\rho)$ are almost all zero (their precise definition is given below).

Lemma 2.2. *We have*

$$n_p(\rho \otimes \chi) = n_p(\chi) \dim \rho$$

when $n_p(\chi)$ is sufficiently large.

Proof. Fix a character χ of conductor p^n ; we know that χ factors through $\text{Gal}(K_n/\mathbb{Q})$ where $K_n = \mathbb{Q}(\mu_{p^n})$. As ρ is an Artin representation, ρ factors through $\text{Gal}(L/\mathbb{Q})$ for some finite extension L/\mathbb{Q} . We may regard $\rho \otimes \chi$ as a representation of $\text{Gal}(LK_n/\mathbb{Q})$.

Let G be the decomposition group of LK_n/\mathbb{Q} at p , and let $G_i = G_i((LK_n)_{\mathfrak{p}_n}/\mathbb{Q}_p)$ be the i -th ramification group, where \mathfrak{p}_n denotes a prime of LK_n above p . By definition of the Artin conductor (see [9, Chapter VII]) we have

$$n_p(\rho \otimes \chi) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} \left(\dim \rho - \frac{1}{g_i} \sum_{s \in G_i} \text{Tr } \rho(s) \chi(\rho) \right)$$

where $g_i = |G_i|$. We observe that

$$\frac{1}{g_i} \sum_{s \in G_i} \chi(s) \text{Tr}(\rho)$$

is the inner product of the characters ρ and $\overline{\chi}$ (which is defined in [12]). This equals the number of copies of $\overline{\chi}$ in ρ , as representations restricted to G_i . Let i_ρ be the smallest integer such that $G_i \subseteq \ker \rho$ for $i \geq i_\rho$. By Lemma 2.1 we know that i_ρ is independent of n , so by taking n sufficiently large we can

assume that χ is non-trivial on G_{i_ρ} . Therefore if $i < i_\rho$, the sum above must be zero as ρ restricted to G_{i_ρ} is a sum of copies of the trivial representation, and χ is non-trivial. Also, if $i \geq i_\rho$ then $\text{Tr } \rho(s) = \dim \rho$ for any $s \in G_i$ by the definition of i_ρ . Putting this into the formula above we deduce that $n_p(\rho \otimes \chi) = n_p(\chi) \dim \rho$. \square

3 The approximate functional equation

Let E be an elliptic curve and ρ an Artin representation, both defined over \mathbb{Q} . The L -series of E twisted by ρ is defined by setting

$$L(E, \rho, s) = \prod_{q \text{ prime}} P_q(E, \rho, q^{-s})^{-1}$$

where the local polynomial at the prime q is given by

$$P_q(E, \rho, T) = \det \left(1 - \text{Frob}_q^{-1} \cdot T | (H_l^1(E) \otimes_{\overline{\mathbb{Q}}_l} V_{\rho, l}) \right).$$

where l is a prime different from q . Here, $H_l^1(E)$ is the dual of the l -adic Tate module of E , and $V_{\rho, l}$ is the representation obtained from ρ by extending scalars to $\overline{\mathbb{Q}}_l$. This Euler product converges only on the right half-plane $\text{Re}(s) > 3/2$ but conjecturally it may be continued to a holomorphic function on the whole complex plane. Let us put

$$L_\infty(s) = (2(2\pi)^{-s} \Gamma(s))^{\dim \rho}$$

and define the completed L -function

$$\widehat{L}(E, \rho, s) := L_\infty(s) L(E, \rho, s).$$

We also write $N(E, \rho)$ for the conductor associated to the twist of E by ρ . Then, we have the following conjecture (see [13] or [4, §2] for example):

Conjecture 3.1. *The completed L -function $\widehat{L}(E, \rho, s)$ has an analytic continuation to the whole complex plane, and satisfies the functional equation*

$$\widehat{L}(E, \rho, s) = w(E, \rho) N(E, \rho)^{1-s} \widehat{L}(E, \rho^*, 2-s)$$

where ρ^* is the contragredient representation to ρ , and the root number $w(E, \rho)$ is a complex number of absolute value 1.

Suppose ρ is an Artin representation for which Conjecture 3.1 is satisfied. Let

$$L(E, \rho, s) = \sum_{n \geq 1} c_n n^{-s}$$

be the Dirichlet series expression which is valid for $\text{Re}(s) > 3/2$. For $\beta \in \mathbb{C}$ and positive $u \in \mathbb{R}$ we define the function

$$F_\beta(u) := \frac{1}{2\pi i} \int_{\text{Re}(s)=2} L_\infty(s + \beta) u^{-s} \frac{ds}{s}.$$

Now we assume that $1 \leq \operatorname{Re}(\beta) \leq 3/2$. Using Cauchy's theorem and the functional equation, we obtain the following formula (the approximate functional equation):

$$\widehat{L}(E, \rho, \beta) = \sum_{n \geq 1} c_n n^{-\beta} F_{\beta}(ny) + w N^{1-\beta} \sum_{n \geq 1} c_n^* n^{\beta-2} F_{2-\beta}\left(\frac{n}{Ny}\right) \quad (1)$$

for $y > 0$, where $N = N(E, \rho)$ and $w = w(E, \rho)$. We also note that the function $F_{\beta}(u)$ has the following properties:

$$u^k F_{\beta}(u) \rightarrow 0 \quad \text{as } u \rightarrow \infty$$

for any $k \geq 0$, and

$$F_{\beta}(u) \rightarrow L_{\infty}(\beta) \quad \text{as } u \rightarrow 0.$$

We can check these by shifting the line of integration to $\operatorname{Re}(s) = u^{k+\theta}$ and $\operatorname{Re}(s) = u^{-\theta}$ respectively, for θ small and positive. Further, one can show that

$$F_{\beta}(u) \ll 1 + u^{\theta}$$

as u approaches zero, for any small $\theta > 0$. These statements also hold for $F_{2-\beta}(u)$.

4 Proof of Theorem 1.1

First, we observe that it suffices to prove Theorem 1.1 in the case $S = \{p\}$ for a prime p at which E has good reduction. To see this, we take a Dirichlet character χ defined modulo pm , where $(m, p) = 1$. Such a character may be written as a product $\chi = \chi_1 \chi_2$ for characters χ_1 modulo p and χ_2 modulo m , and considering the characters as one-dimensional Artin representations we have $\rho \otimes \chi = (\rho \otimes \chi_2) \otimes \chi_1$. Therefore we apply the result for $S = \{p\}$ to the representation $\rho \otimes \chi_2$, which implies the existence of infinitely many twists unramified outside $\{p\} \cup \{\text{primes } q \mid m\}$ with the claimed non-vanishing property.

We now fix a prime p at which E has good reduction; we will prove the result for characters of p -power conductor. We fix the Artin representation ρ and let χ be a primitive Dirichlet character modulo p^a . Taking $a = n_p(\chi)$ to be sufficiently large, by Lemma 2.2 we know that $n_p(\rho \otimes \chi) = a \dim \rho$. As we assumed E has good reduction at p , we have

$$N(E, \rho \otimes \chi) = N p^{2a \dim \rho}.$$

where N is the prime-to- p part of the conductor, which will remain fixed.

We consider the twisted L -series

$$L(E, \rho \otimes \chi, s) = \sum_{n \geq 1} \chi(n) c_n n^{-s}.$$

In order to apply the approximate functional equation, we assume that it has the analytic continuation and functional equation specified in Conjecture 3.1 for every Dirichlet character χ of conductor p^a (which is hypothesised in Theorem

1.1).

By the functional equation, it suffices to consider the values of $L(E, \rho \otimes \chi, s)$ in the right half of the critical strip. Let $\beta \in \mathbb{C}$ satisfy $1 \leq \operatorname{Re}(\beta) \leq 3/2$; we apply equation (1) to write

$$\begin{aligned} \widehat{L}(E, \rho \otimes \chi, \beta) &= \sum_{n \geq 1} \chi(n) c_n n^{-\beta} F_{\beta} \left(\frac{ny}{Np^{2ad}} \right) \\ &+ w(E, \rho, \chi) (Np^{2ad})^{1-\beta} \sum_{n \geq 1} \overline{\chi}(n) c_n^* n^{\beta-2} F_{2-\beta} \left(\frac{n}{y} \right) \end{aligned}$$

for $y > 0$; here we have re-normalised the variable y via $y \mapsto yN^{-1}p^{-2ad}$. We observe that the χ -twist does not change the Gamma-factor $L_{\infty}(s)$ so it does not affect the function F .

We now average over primitive characters modulo p^a : we define

$$A(n) := \sum_{\chi \bmod p^a}^* \chi(n) F_{\beta} \left(\frac{ny}{Np^{2ad}} \right)$$

and

$$B(n) := \sum_{\chi \bmod p^a}^* w(E, \rho, \chi) \overline{\chi}(n) p^{2ad(1-\beta)}$$

where the symbol \sum^* denotes the sum over primitive characters only. Then we have

$$\begin{aligned} \sum_{\chi \bmod p^a}^* \widehat{L}(E, \rho \otimes \chi, \beta) &= \sum_{n \geq 1} A(n) c_n n^{-\beta} \\ &+ N^{1-\beta} \sum_{n \geq 1} B(n) c_n^* n^{\beta-2} F_{2-\beta} \left(\frac{n}{y} \right). \end{aligned}$$

We will proceed to show that this sum tends to infinity with a when

$$\frac{3}{2} - \frac{2}{2d+1} < \operatorname{Re}(\beta) \leq \frac{3}{2}$$

which will establish Theorem 1.1. We write the sum above in the form

$$\sum_{\chi \bmod p^a}^* \widehat{L}(E, \rho \otimes \chi, \beta) = A(1) + \Sigma_1 + \Sigma_2$$

where we have put

$$\Sigma_1 = \sum_{n \geq 2} A(n) c_n n^{-\beta} \quad \text{and} \quad \Sigma_2 = N^{1-\beta} \sum_{n \geq 1} B(n) c_n^* n^{\beta-2} F_{2-\beta} \left(\frac{n}{y} \right).$$

We will show that the parameter y may be specified so that $|A(1)| \gg |\Sigma_1|$ and $|A(1)| \gg |\Sigma_2|$ as $a \rightarrow \infty$ which will prove the result.

5 Estimating the sums

We introduce a parameter x such that $xy = p^{2da}$ and assume both x and y are fixed positive powers of p^a (we will discuss how they can be specified in §6). We have

$$A(1) = \sum_{\chi \bmod p^a}^* F_\beta \left(\frac{y}{Np^{2ad}} \right).$$

By our choice of x and y we have

$$\frac{y}{Np^{2ad}} = \frac{1}{Nx} \rightarrow 0$$

as $a \rightarrow \infty$. Recalling that $F_\beta(u)$ tends to the non-zero constant $L_\infty(\beta)$ as $u \rightarrow 0$ we have

$$|A(1)| \sim \sum_{\chi \bmod p^a}^* 1$$

and therefore $|A(1)| \gg p^a$ (recalling that p is constant and a is tending to infinity). Next we consider Σ_1 ; we have

$$|A(n)| = \left| F_\beta \left(\frac{ny}{Np^{2ad}} \right) \sum_{\chi \bmod p^a}^* \chi(n) \right|.$$

Suppose first that $n \leq x^{1+\epsilon}$ for ϵ small and positive. Using the estimate $F_\beta(u) \ll 1 + u^\theta$, we obtain

$$|A(n)| \ll x^\epsilon \left| \sum_{\chi \bmod p^a}^* \chi(n) \right|.$$

By basic properties of character sums we have

$$\sum_{\chi \bmod m}^* \chi(n) = \sum_{b|(n-1, m)} \varphi(b) \mu \left(\frac{m}{b} \right)$$

if $(n, m) = 1$ (see [7, 3.8]). From this we infer that the character sum factor in $|A(n)|$ is $\ll p^a$ if $n-1$ is divisible by p^{a-1} , and zero otherwise. The Dirichlet coefficients c_n are known to satisfy $|c_n| \leq n^{1/2+\epsilon}$ for any $\epsilon > 0$, and putting these facts together we get

$$\left| \sum_{2 \leq n \leq x^{1+\epsilon}} A(n) c_n n^{-\beta} \right| \ll p^a x^\epsilon \sum_{2 \leq n \leq x^{1+\epsilon}}^\dagger n^{1/2 - \operatorname{Re}(\beta) + \epsilon} \quad (2)$$

where \sum^\dagger denotes the sum over integers congruent to 1 mod p^{a-1} only. We then observe that

$$\begin{aligned} \sum_{2 \leq n \leq x^{1+\epsilon}}^\dagger n^{-\theta} &= \sum_{1 \leq r \leq x^{1+\epsilon} p^{1-a}} (1 + rp^{a-1})^{-\theta} \\ &\ll p^{-a\theta} (x^{1+\epsilon} p^{1-a})^{1-\theta+\epsilon} \sum_{1 \leq r} r^{-1-\epsilon} \\ &\ll p^{-a+\epsilon} x^{1-\theta+\epsilon} \end{aligned}$$

for any constant $\theta > 0$. Putting $\theta = \operatorname{Re}(\beta) - 1/2 - \epsilon$ and combining this with (2) we deduce

$$\left| \sum_{2 \leq n \leq x^{1+\epsilon}} A(n) c_n n^{-\beta} \right| \ll x^{3/2 - \operatorname{Re}(\beta) + \epsilon}.$$

We must deal with the terms for $n > x^{1+\epsilon}$; however the fact that $u^k F_\beta(u) \rightarrow 0$ as $u \rightarrow \infty$ for any $k \geq 0$ shows that the contribution of these terms is negligible. We conclude that

$$|\Sigma_1| \ll x^{3/2 - \operatorname{Re}(\beta) + \epsilon}. \quad (3)$$

We now consider the sum

$$\Sigma_2 = N^{1-\beta} \sum_{n \geq 1} B(n) c_n^* n^{\beta-2} F_{2-\beta} \left(\frac{n}{y} \right).$$

As above, it will suffice to bound the terms with $n \leq y^{1+\epsilon}$ as the tail will become constant due to the rapid decay of $F_{2-\beta}(u)$ as $u \rightarrow \infty$. By the estimates for $F_{2-\beta}(u)$ in §3 we have the bound

$$F_{2-\beta}(u) \ll \frac{1+u^\epsilon}{1+u^2}$$

for any small $\epsilon > 0$. Following Luo's proof from [8], we apply Cauchy's inequality and split up the product to get

$$\begin{aligned} \left| \sum_{1 \leq n \leq y^{1+\epsilon}} B(n) c_n^* n^{\beta-2} F_{2-\beta} \left(\frac{n}{y} \right) \right| &\ll \sum_{1 \leq n \leq y^{1+\epsilon}} |B(n) c_n^*| n^{\operatorname{Re}(\beta)-2} \frac{1 + \left(\frac{n}{y}\right)^\epsilon}{1 + \left(\frac{n}{y}\right)^2} \\ &\ll \left(\sum_{n \leq y^{1+\epsilon}} |c_n^*|^2 n^{2(\operatorname{Re}(\beta)-2)} \left(1 + \left(\frac{n}{y}\right)^\epsilon\right) \right)^{1/2} \left(\sum_{n \leq y^{1+\epsilon}} |B(n)|^2 \left(1 + \left(\frac{n}{y}\right)^2\right)^{-1} \right)^{1/2} \\ &\ll y^{\operatorname{Re}(\beta)-1+\epsilon} \left(\sum_{n \leq y^{1+\epsilon}} |B(n)|^2 \left(1 + \left(\frac{n}{y}\right)^2\right)^{-1} \right)^{1/2}. \end{aligned}$$

Here we have bounded the first factor in the product above by a method analogous to that used for Σ_1 . Let us define

$$H(u) := \frac{1}{\pi(1+u^2)}$$

which is the Fourier transform of $e^{-2\pi|u|}$. By the estimate above we have

$$|\Sigma_2| \ll y^{\operatorname{Re}(\beta)-1+\epsilon} \left(\sum_{n \in \mathbb{Z}} |B(n)|^2 H \left(\frac{n}{y} \right) \right)^{1/2}.$$

Here we have increased the range of the sum in order to apply Poisson summation later. Recall that

$$B(n) = \sum_{\chi \bmod p^a}^* w_\chi \overline{\chi}(n) p^{2ad(1-\beta)},$$

where we have written w_χ for the root number $w(E, \rho \otimes \chi)$. We have

$$\sum_{n \in \mathbb{Z}} |B(n)|^2 H\left(\frac{n}{y}\right) \leq \sum_{\chi \bmod p^a}^* \sum_{\psi \bmod p^a}^* p^{4ad(1-\operatorname{Re}(\beta))} \left| w_\chi \overline{w_\psi} \sum_{n \in \mathbb{Z}} \overline{\chi} \psi(n) H\left(\frac{n}{y}\right) \right|.$$

First we consider the diagonal terms in this sum: those with $\chi = \psi$. For these terms we get

$$\sum_{\chi \bmod p^a}^* p^{4ad(1-\operatorname{Re}(\beta))} \left| w_\chi w_{\overline{\chi}} \sum_{n \in \mathbb{Z}} H\left(\frac{n}{y}\right) \right| \ll p^{4ad(1-\operatorname{Re}(\beta))+a} \sum_{n \in \mathbb{Z}} H\left(\frac{n}{y}\right)$$

recalling that $|w_\chi| = 1$. By the Poisson summation formula we have

$$\sum_{n \in \mathbb{Z}} H\left(\frac{n}{y}\right) = y \sum_{h \in \mathbb{Z}} T(yh)$$

where $T(u) = e^{-2\pi|u|}$. All terms in this sum decay exponentially with y , except that for $h = 0$. So $y \sum_{h \in \mathbb{Z}} T(yh) \ll y$ which implies that the diagonal terms are

$$\ll y p^{4ad(1-\operatorname{Re}(\beta))+a}.$$

Now we consider the terms with $\chi \neq \psi$. By twisted Poisson summation we have

$$\sum_{n \in \mathbb{Z}} \overline{\chi} \psi(n) H\left(\frac{n}{y}\right) = \frac{y}{\tau(\chi \overline{\psi})} \sum_{h \in \mathbb{Z}} \chi \overline{\psi}(h) T\left(\frac{yh}{N_{\chi \overline{\psi}}}\right).$$

Therefore the off-diagonal terms in the character sum are bounded above by

$$\sum_{\chi}^* \sum_{\psi \neq \chi}^* p^{4a(1-\operatorname{Re}(\beta))} \left| w_\chi \overline{w_\psi} \frac{y}{\tau(\chi \overline{\psi})} \sum_{h \in \mathbb{Z}} \chi \overline{\psi}(h) T\left(\frac{yh}{N_{\chi \overline{\psi}}}\right) \right|.$$

In this case, the $\chi \overline{\psi}$ is non-trivial and so vanishes at zero; all other terms in the sum over h decay exponentially with y . Therefore the absolute value of this sum is

$$p^{4ad(1-\operatorname{Re}(\beta))} \sum_{\chi}^* \sum_{\psi \neq \chi}^* \frac{1}{|\tau(\chi \overline{\psi})|}$$

again using the fact that $|w_\chi| = 1$. For any χ primitive modulo p^a and any $0 \leq c < a$, there are $\varphi^*(p^c)$ characters ψ which are primitive modulo p^a such that $\chi \overline{\psi}$ has conductor p^c ; here we have written $\varphi^*(p^c)$ for the number of primitive characters modulo p^c . Using this fact, we deduce

$$\sum_{\chi}^* \sum_{\psi}^* \frac{1}{|\tau(\chi \overline{\psi})|} \ll \sum_{\chi}^* \sum_{c=1}^a \frac{\varphi^*(p^c)}{p^{c/2}} \ll p^{3a/2}$$

from which we get

$$\sum_{n \in \mathbb{Z}} |B(n)|^2 H\left(\frac{n}{y}\right) \ll p^{4da(1-\operatorname{Re}(\beta))} (y p^a + p^{3a/2}).$$

Putting this into the earlier bound for Σ_2 obtained from Cauchy's inequality, we obtain

$$|\Sigma_2| \ll y^{\operatorname{Re}(\beta)-1+\epsilon} p^{2da(1-\operatorname{Re}(\beta))} (y^{1/2} p^{a/2} + p^{3a/4}). \quad (4)$$

6 Conclusion

Let us write $p^a = P$ for our parameter which is tending to infinity. We will now specify x and y : we require them to satisfy $xy = P^{2d}$ and to tend to infinity with P , so we may write

$$y = P^{2d\gamma} \quad \text{and} \quad x = P^{2d(1-\gamma)}$$

for $0 < \gamma < 1$. Putting $\text{Re}(\beta) = \sigma$, we may write estimates (3) and (4) as follows:

$$\begin{aligned} |\Sigma_1| &\ll P^{2d(1-\gamma)(3/2-\sigma+\epsilon)} \\ \text{and } |\Sigma_2| &\ll P^{2d\gamma(\sigma-1+\epsilon)+2d(1-\sigma)}(P^{d\gamma+1/2} + P^{3/4}) \end{aligned}$$

Recal that we have $|A(1)| \gg P$, so $A(1)$ will eventually grow more rapidly than Σ_1 and Σ_2 if the following three inequalities hold:

$$2d(1-\gamma)(3/2-\sigma) < 1, \tag{5}$$

$$2d\gamma(\sigma-1/2) + 2d(1-\sigma) < 1/2, \tag{6}$$

$$2d\gamma(\sigma-1) + 2d(1-\sigma) < 1/4. \tag{7}$$

Here, inequalities (6) and (7) come from the two terms in the estimate (4). Observe that we have neglected ϵ which may be chosen as small as we like.

If $\sigma = 3/2$, one checks easily that these inequalities are satisfied if γ is chosen to be sufficiently small, so we now assume that $1 \leq \sigma < 3/2$. From inequalities (5) and (6) we get an upper and lower bound on γ . One checks easily that these are consistent with $0 < \gamma < 1$; it remains to check for which σ they can be made consistent with each other. Combining them, we obtain

$$\frac{6d-1}{4d+2} < \sigma$$

which (given the functional equation) is precisely the hypothesis imposed in Theorem 1.1. Rearranging inequality (7) above, we get another upper bound for γ which places a restriction on σ ; however it is straightforward to show that this condition is weaker than the one above.

We conclude that, when $\sigma = \text{Re}(\beta)$ satisfies the inequality above, we can choose our parameter γ so that $|A(1)|$ dominates $|\Sigma_1|$ and $|\Sigma_2|$ as $a \rightarrow \infty$. As stated in §4 this completes the proof of theorem 1.1.

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